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Fractional-Order Diffusion-Wave Equation

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The fractional-order diffusion-wave equation is an evolution equation of order $\alpha \in (0, 2]$ which continues to the diffusion equation when $\alpha \to 1$ and to the wave equation when $\alpha \to 2$. We prove some properties of its solution and give some examples. We define a new fractional calculus (negative-direction fractional calculus) and study some of its properties. We study the existence, uniqueness, and properties of the solution of the negative-direction fractional diffusion-wave problem.

1. INTRODUCTION

Let X be a Banach space, let $Y \subseteq X$, and let $L_1(I, Y)$ be the set of functions $f(t) \in Y$ (for each $t \in I$) which is integrable on I (where $I = [0, T]$, $T < \infty$). Let A be a closed linear operator defined on X with domain $D(A)$ dense in X. We have d^α/dt^α as the fractional differential operator of order $\alpha > 0$ (El-Sayed, 1992, 1993, 1995; El-Sayed and Ibrahime, n.d.; Gelfand and Shilove, 1958).

Consider the fractional evolution equation

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t), \qquad t > 0, \quad 0 < \alpha \leq 2 \tag{1.1}$$

Here we study first the continuation of equation (1.1) when $\alpha \to 1$ and $\alpha \to 2$. We define the fractional diffusion-wave equation and the fractional diffusion-wave problem and prove that it is well posed. Combining our results here and the results of Wyss (1986), Schnrider and Wyss (1989), and Mainardi (1994), we give some examples.

Second, we define the negative-direction fractional calculus (integral and derivative) and prove some of its properties.

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Finally, we study the existence, uniqueness, and properties (concerning continuation backward in time) of the solution of the negative-direction fractional diffusion-wave problem.

2. DIFFUSION-WAVE EQUATION

Consider the two initial value problems

$$(\mathrm{P}) \qquad \begin{cases} \dfrac{d^{\gamma}u(t)}{dt^{\gamma}} = Au(t), & t > 0, \quad 0 < \gamma \le 1 \\ u(0) = u_0 \end{cases}$$

$$(\mathrm{Q}) \qquad \begin{cases} \dfrac{d^{\beta}u(t)}{dt^{\beta}} = Au(t), & t > 0, \quad 1 < \beta \le 2 \\ u(0) = u_0, \qquad u_t(0) = u_1 \end{cases}$$

where A satisfies the following condition:

(I) A generates an analytic semigroup $\{T(t),\ t \ge 0\}$ on X; in particular, the resolvent set of A contains the set

$$\Lambda_1 = \left\{\lambda \in \mathrm{C}\colon |\arg \lambda| < \frac{\pi}{2} + \delta_1\right\}, \qquad 0 < \delta_1 < \frac{\pi}{2}$$

and $\|(\lambda I - A)^{-1}\| \le M/|\lambda|$, Re $\lambda > 0$ on Λ_1 for some constant $M > 0$, and

$$\frac{d^{\alpha}u(t)}{dt^{\alpha}} = \frac{d^{k}u(t)}{dt^{k}} * \phi_{k-\alpha}(t), \qquad k - 1 = [\alpha] \tag{2.1}$$

is the fractional derivative of order $\alpha > 0$ of the differentiable function $u(t)$, where for $\alpha > 0$, $\phi_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t > 0$, and $\phi_\alpha(t) = 0$ for $t \le 0$ [for the properties of $\phi_\alpha(t)$ see Gelfand and Shilove (1958)].

The following results are proved in El-Sayed (1995).

Theorem 2.1. Let u_1 and $u_0 \in D(A^2)$. If A satisfies (I), then there exists a unique solution $u_\gamma(t) \in L_1(I, D(A))$ of the problem (P) given by

$$u_\gamma(t) = u_0 - \int_0^t r_\gamma(s)e^{s}u_0\, ds, \qquad 0 < \gamma \le 1 \tag{2.2}$$

and a unique solution $u_\beta(t) \in L_1(I, D(A))$ of problem (Q) given by

$$u_\beta(t) = u_0 + tu_1 - \int_0^t r_\beta(s)e^{s}(u_0 + (t - s)u_1)\, ds \tag{2.3}$$

and when $u_1 = 0$, we have

$$\lim_{\gamma\to 1^-} u_\gamma(t) = \lim_{\beta\to 1^+} u_\beta(t) = T(t)u_0 = u_1(t)$$

$$\lim_{\gamma\to 1^-} \frac{d^\gamma u_\gamma(t)}{dt^\gamma} = \lim_{\beta\to 1^+} \frac{d^\beta u_\beta(t)}{dt^\beta} = AT(t)u_0 = \frac{du_1}{dt}$$

where $u_1(t)$ is the solution of the Cauchy problem

$$\frac{du(t)}{dt} = Au(t), \qquad u(0) = u_0 \tag{2.4}$$

and $r_\alpha(t) \in L(D(A^2), D(A)) \cap L(D(A), X)$ is the resolvent operator satisfying [for $y \in D(A^2)$]

$$\begin{aligned} r_\alpha(t)y &= -e^{-t}\phi_\alpha(t)Ay + r_\alpha(t) * e^{-t}\phi_\alpha(t)Ay \\ &= e^{-t}\phi_\alpha(t)Ay + r_\alpha(t)Ay * e^{-t}\phi_\alpha(t) \end{aligned}$$

[for other properties of $r_\alpha(t)$ see El-Sayed (1995)].

Now let $X = R^k$ and $u(x, t)$: $R^n \times I \to R^k$; then we have the following definition.

Definition 2.2. The fractional D-W (diffusion-wave) equation is the equation

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = Au(x, t), \qquad t > 0, \quad 0 < \alpha \le 2 \tag{2.5}$$

and the fractional diffusion-wave problem is the Cauchy problem

$$\text{(D-W)} \qquad \begin{cases} \dfrac{\partial^\alpha u(x, t)}{\partial t^\alpha} = Au(x, t), & t > 0, \quad x \in R^n, \quad 0 < \alpha \le 2 \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = 0, & x \in R^n \end{cases}$$

Lemma 2.3. Let $u(x, t)$: $R^n \times I \to R^n$ and $0 < \alpha \le 2$. If $u_t(x, 0) = 0$, then as $\alpha \to 1$ the D-W problem reduces to the diffusion problem

$$\frac{\partial u(x, t)}{\partial t} = Au(x, t), \qquad u(x, 0) = u_0(x), \quad x \in R^n, \quad t > 0 \tag{2.6}$$

and as $\alpha \to 2$, the D-W problem reduces to the wave problem

$$\frac{\partial^2 u(x, t)}{\partial^2 t} = Au(x, t), \qquad u(x, 0) = u_0(x), \quad u_t(x, 0) = 0 \tag{2.7}$$

Proof. Let $0 < \alpha \leq 1$; then from the properties of the fractional derivative and the convolution operation (Gelfand and Shilove, 1958) we get

$$\lim_{\alpha \to 1^-} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \lim_{\alpha \to 1} \left(\frac{\partial u(x, t)}{\partial t} * \phi_{1-\alpha}(t) \right)$$

$$= \frac{\partial u(x, t)}{\partial t} * \delta(t) = \frac{\partial u(x, t)}{\partial t}$$

and when $1 \leq \alpha < 2$ we have

$$\lim_{\alpha \to 1^+} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \lim_{\alpha \to 1} \left(\frac{\partial^2 u(x, t)}{\partial t^2} * \phi_{2-\alpha}(t) \right)$$

$$= \lim_{\alpha \to 1} \left(\frac{\partial^2 u(x, t)}{\partial t^2} * \phi_1(t) \right)$$

$$= \frac{\partial u(x, t)}{\partial t} - u_t(x, 0) = \frac{\partial u(x, t)}{\partial t}$$

we also have

$$\lim_{\alpha \to 2} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \lim_{\alpha \to 2} \left(\frac{\partial^2 u(x, t)}{\partial t^2} * \phi_{2-\alpha}(t) \right)$$

$$= \frac{\partial^2 u(x, t)}{\partial t^2} * \delta(t) = \frac{\partial^2 u(x, t)}{\partial t^2}$$

and the result follows.

Combining the results of Theorem 2.1 and Lemma 2.3, we can easily prove the following theorem.

Theorem 2.4. Let $u_0(x) \in D(A^2)$. If A satisfies the condition (I) where $X = R^k$, then the D-W problem has a unique solution $u_\alpha(x, t) \in L_1(I, D(A))$ continuous with respect to $\alpha \in (0, 2]$ and satisfies

$$\lim_{\alpha \to 1^-} u_\alpha(x, t) = \lim_{\alpha \to 1^+} u_\alpha(x, t) = T(t)u_0(x) \tag{2.8}$$

$$\lim_{\alpha \to 2^-} u_\alpha(x, t) = u_2(x, t) \tag{2.9}$$

where $u_2(x, t)$ is the solution of the wave problem (2.7).

Theorem 2.5. If the assumptions of Theorem 2.4 are satisfied, then the solution of the D-W problem continuously depends on the initial data $u_0(x)$.

Proof. Let $u_\alpha(x, t)$ be the solution of the D-W problem and $\tilde{u}_\alpha(x, t)$ be the solution of the D-W problem with $\tilde{u}_0(x)$ instead of $u_0(x)$; then for given $\epsilon > 0$ such that $\|u_0(x) - \tilde{u}_0(x)\|_{D(A^2)} < \epsilon$ we have (El-Sayed, 1995)

$$u_\alpha(x, t) - \tilde{u}_\alpha(x, t)$$
$$= u_0(x) - \tilde{u}_0(x) + \int_0^t r_\alpha(s) e^s (u_0(x) - \tilde{u}_0(x))\, ds$$

and so

$$\|u_\alpha(x, t) - \tilde{u}_\alpha(x, t)\|_{D(A)}$$
$$\leq \left(1 + \frac{ce^t t^\alpha}{\alpha}\right) \|u_0(x) - \tilde{u}_0(x)\|_{D(A^2)}$$
$$\leq \left(1 + \frac{ce^t t^\alpha}{\alpha}\right) \epsilon = \delta(\epsilon)$$

which proves the theorem.

3. EXAMPLES

1. Let $A = (-1)^{1+m} \nabla^{2m}$, $x = R^k$, and $u_0(x) \in W_p^{4m}(R^n)$, $1 \leq p < \infty$. Then the results here can be applied to the D-W problem

$$\text{(D-W)} \quad \begin{cases} \dfrac{\partial^\alpha u(x, t)}{\partial t^\alpha} = (-1)^{1+m}\, \nabla^{2m} u(x, t) \\ \qquad\qquad x \in R^n, \quad t > 0, \quad 0 < \alpha \leq 2 \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = 0 \end{cases}$$

2. Let $m = 1$, $k = 1$, and $p = 2$ in Example 1. Then combining our results here and the results of Wyss (1986) and Schnrider and Wyss (1989), we deduce that the D-W problem

$$\begin{cases} \dfrac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \nabla^2 u(x, t), \qquad t > 0, \quad x \in R^n, \quad 0 < \alpha \leq 2 \\ u(x, 0) = u_0(x) \in W_2^4(R^n), \qquad u_t(x, 0) = 0 \end{cases}$$

has a unique solution $u_\alpha(x, t) \in L_1(I, W_2^2(R^n))$ given by

$$u_\alpha(x, t) = \int_{R^n} G_\alpha(x - y, t) u_0(y)\, dy, \qquad 0 < \alpha \leq 2$$

where $G_\alpha(x, t)$ is the inverse Fourier transform of the Mittag–Leffler function

$$F_\alpha(\rho^2 t^\alpha) = \sum_{j=0}^{\infty} \frac{(-\rho^2 t^\alpha)^j}{\Gamma(1 + j\alpha)}, \qquad \rho^2 = \rho \cdot \rho, \quad \rho \in R^n$$

and

$$\lim_{\alpha \to 1} u_\alpha(x, t) = \int_{R^n} G_1(x - y, t) u_0(y)\, dy$$

where

$$G_1(x, t) = \frac{1}{[2(\pi t)^{1/2}]^n} \exp\left(\frac{-|x|^2}{4t}\right)$$

and the solution of the wave problem

$$\begin{cases} \dfrac{\partial^2 u(x, t)}{\partial t^2} = \nabla^2 u(x, t), & x \in R^n, \quad t > 0 \\ u(x, 0) = u_0(x), & u_t(x, 0) = 0 \end{cases}$$

is given by

$$u_2(x, t) = \lim_{\alpha \to 2} \int_{R^n} G_\alpha(x - y, t) u_0(y)\, dy$$

When $n = 1$, these results ensure the validity of the results in Mainardi (1994).

3. Let

$$D(A) = \{u \in L_2(\Omega) \colon \nabla^2 u \in L_2(\Omega),\, u|_{\partial\Omega} = 0\}, \qquad Au = \nabla^2 u$$

Then our results here can be applied to the D-W problem

$$\begin{cases} \dfrac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \nabla^2 u(x, t), & t > 0, \quad x \in \Omega \subset R^n, \quad 0 < \alpha \le 2 \\ u(x, t) = 0, & x \in \partial\Omega, \quad t \ge 0 \\ u(x, 0) = u_0(x) \in W_2^4(\Omega), & u_t(x, 0) = 0, \quad X \in \Omega \end{cases}$$

4. NEGATIVE-DIRECTION FRACTIONAL CALCULUS

Let $C^n(J, X)$ be the set of continuous functions with continuous derivatives up to order n defined on $J = (-\infty, 0)$ with values in X, and $\mathbf{C}^n$ be the class of functions $f(t) \in C^n(J, X)$ with $D^j f(t)|_{t=0} = 0, j = 0, 1, \ldots, n - 1$.

We propose a new fractional calculus [defined on $J = (-\infty, 0)$] based on the generalized function $\Psi_\alpha(t)$ (Gelfand and Shilove, 1958), $t \in J$ and $\alpha > 0$, where

$$\Psi_\alpha(t) = \begin{cases} 0 & t > 0 \\ \dfrac{|t|^{\alpha-1}}{\Gamma(\alpha)}, & t \le 0 \end{cases}$$

Definition 4.1. The negative-direction fractional integral of order $\alpha > 0$ of the integrable function $f(t)$ is defined by

$$S^{-\alpha}f(t) = \Psi_\alpha(t) * f(t) = \int_t^0 \frac{(\theta - t)^{\alpha-1}}{\Gamma(\alpha)} f(\theta)\, d\theta, \qquad t < 0 \tag{4.1}$$

and the fractional derivative of order $\alpha > 0$, $[\alpha] = n - 1$, of the function $g(t) \in C^n(J, X)$ is defined by

$$S^\alpha g(t) = S^{-(n-\alpha)}(-1)^n D^n g(t) = (-1)^n \Psi_{n-\alpha}(t) * D^n g(t) \tag{4.2}$$

Lemma 4.1. Let $\alpha > 0$. If $f(t) \in C(J, X)$, then

$$\lim_{\alpha \to p} S^{-\alpha}f(t) = S^{-p}f(t), \qquad p = 1, 2, \ldots \tag{4.3}$$

where

$$S^{-1}f(t) = \int_t^0 f(\theta)\, d\theta \tag{4.4}$$

Also, if $g(t) \in \mathbf{C}^n$, then

$$D^n S^{-\alpha} g(t) = S^{-\alpha} D^n g(t) \tag{4.5}$$

$$S^{-\alpha} g(t) = S^{-(n+\alpha)}(-1)^n D^n g(t) \tag{4.6}$$

Proof. Let $\|\cdot\|$ be the norm of X. From Definition 4.1 we have

$$\begin{aligned} &\|S^{-\alpha}f(t) - S^{-p}f(t)\| \\ &= \left\| \int_t^0 \left(\frac{(\theta - t)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\theta - t)^{p-1}}{(p-1)!} \right) f(\theta)\, d\theta \right\| \\ &\le \sup_t \|f(t)\| \int_t^0 \left| \left(\frac{(\theta - t)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\theta - t)^{p-1}}{(p-1)!} \right) \right| d\theta \end{aligned}$$

but since $x^{\alpha-1}/\Gamma(\alpha) \to x^{p-1}/(p-1)!$ as $\alpha \to p$, $p = 1, 2, \ldots$, we get (4.3). Now since (integrating by parts)

$$S^{-\alpha}g(t) = \int_t^0 \frac{(\theta - t)^{\alpha-1}}{\Gamma(\alpha)} g(\theta)\, d\theta$$

$$= \frac{(-t)^{\alpha}}{\Gamma(\alpha+1)} g(0) - \int_t^0 \frac{(\theta - t)^{\alpha}}{\Gamma(\alpha+1)} \left(\frac{d}{d\theta} g(\theta)\right) d\theta$$

then for $g(t) \in \mathbf{C}^1$, we get $S^{-\alpha}g(t) = S^{-(\alpha+1)}(-1)Dg(t)$, $\alpha > 0$. Repeating the process, we get (4.6), and differentiating (4.6), we get (4.5).

Lemma 4.2. Let $\alpha > 0$, $[\alpha] = n - 1$. If $g(t) \in \mathbf{C}^n$, then

$$\lim_{\alpha \to p} S^{\alpha}g(t) = (-1)^p D^p g(t), \qquad p = 0, 1, 2, \ldots, n-1 \tag{4.7}$$

where

$$S^0 g(t) = g(t), \qquad \text{i.e.,} \quad S^0 \text{ is the identity} \tag{4.8}$$

Proof. From Definition 4.1 we have

$$S^{\alpha}g(t) = S^{-(n-\alpha)}(-1)^n D^n g(t)$$

Then

$$\lim_{\alpha \to p} S^{\alpha}g(t) = S^{-(n-p)}(-1)^n D^n g(t)$$

$$= (-1)^p D^p g(t), \qquad p = 0, 1, 2, \ldots, n-1$$

Now we have the following theorem:

Theorem 4.3. The family $S = \{S^{\alpha}; \alpha \in R\}$ is a multiplicative group on $\mathbf{C}^n$.

Proof. Let $\alpha_1, \alpha_2 > 0$; then for $f(t) \in C(J, X)$ we have

$$S^{-\alpha_1}S^{-\alpha_2}f(t) = \int_t^0 \frac{(\theta - t)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \int_\theta^0 \frac{(\tau - \theta)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} f(\tau)\, d\tau\, d\theta$$

which by Dirichlet's formula gives

$$S^{-\alpha_1}S^{-\alpha_2}f(t) = \int_t^0 \frac{(\tau - t)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1 + \alpha_2)} f(\tau)\, d\tau = S^{-(\alpha_1+\alpha_2)}f(t) \tag{4.9}$$

and for $f(t) \in \mathbf{C}^n$, we have ($[\alpha_1] = n_1 - 1$, $[\alpha_2] = n_2 - 1$)

$$S^{\alpha_1}S^{\alpha_2}f(t) = S^{-(n_1-\alpha_1)}(-D)^{n_1}S^{-(n_2-\alpha_2)}(-D)^{n_2}f(t)$$

$$= S^{-(n_1+n_2-\alpha_1-\alpha_2)}(-D)^{n_1+n_2}f(t)$$

$$= S^{\alpha_1+\alpha_2}f(t), \qquad n_1 + n_2 \le n$$

Let $\alpha_1 > \alpha_2$; then

$$\begin{aligned} S^{\alpha_1}S^{-\alpha_2}f(t) &= S^{-(n_1-\alpha_1)}(-D)^{n_1}S^{-\alpha_2}f(t) \\ &= (-D)^{n_1}S^{-(n_1-\alpha_1)}S^{-\alpha_2}f(t) \\ &= (-D)^{n_1}S^{-(n_1-\alpha_1+\alpha_2)}f(t) = S^{\alpha_1-\alpha_2}f(t) \end{aligned}$$

Also, for $\alpha_1 < \alpha_2$, we get $S^{\alpha_1}S^{-\alpha_2}f(t) = S^{-(\alpha_2-\alpha_1)}f(t)$. Therefore we have $S^{\alpha_1}S^{\alpha_2} \in S$ for $\alpha_1, \alpha_2 \in R$.

In the same way we can prove that

$$S^{\alpha_1}(S^{\alpha_2}S^{\alpha_3})f(t) = (S^{\alpha_1}S^{\alpha_2})S^{\alpha_3}f(t)$$

From Lemma 4.2 we have $I = S^0 \in S$.

Now, since for $\alpha \in R^+$, $[\alpha] = n - 1$ and $f(t) \in C^n$, we have

$$S^{-\alpha}S^{\alpha}f(t) = S^{-\alpha+\alpha}f(t) = S^0f(t) = f(t)$$

so

$$(S^{\alpha})^{-1} = S^{-\alpha} \qquad \text{and} \qquad (S^{-\alpha})^{-1} = S^{\alpha}$$

Finally, for $\alpha_1, \alpha_2 \in R$ and $f(t) \in \mathbf{C}^n$, we have

$$S^{\alpha_1}S^{\alpha_2}f(t) = S^{\alpha_1+\alpha_2}f(t) = S^{\alpha_2}S^{\alpha_1}f(t)$$

which completes the proof.

Now, for the relation between the negative-direction fractional calculus and the usual one we have the following theorem:

Theorem 4.4. Let $f(t) = F(\tau)$, $\tau = -t > 0$, and $\alpha > 0$. If $f(t) \in C^n(J, X)$, then

$$S^{\alpha}f(t) = \frac{d^{\alpha}F(\tau)}{d\tau^{\alpha}}$$

Proof. Let $\beta > 0$; then from Definition 4.1 we have

$$\psi_\beta(t) * f(t) = \int_t^0 \frac{(\theta - t)^{\alpha-1}}{\Gamma(\alpha)} f(\theta)\, d\theta$$

Putting $\theta = -s$, we get

$$\psi_\beta(t) * f(t) = \int_0^{-t} \frac{(-t - s)^{\alpha-1}}{\Gamma(\alpha)} F(-s)\, ds$$

and for $\tau = -t > 0$, we deduce that

$$\psi_\beta(t) * f(t) = \phi_\beta(\tau) * F(\tau) \tag{4.10}$$

Now for $\alpha > 0$, $[\alpha] = n - 1$, we have

$$S^{\alpha}f(t) = S^{-(n-\alpha)}\left(\frac{-d}{dt}\right)^n f(t)$$

$$= \psi_{(n-\alpha)}(t) * \left(\frac{-d}{dt}\right)^n f(t)$$

Using (4.10), we get

$$S^{\alpha}f(t) = \phi_{n-\alpha}(\tau) * \frac{d^nF(\tau)}{d\tau^n} = \frac{d^{\alpha}F(\tau)}{d\tau^{\alpha}}$$

Corollary 4.5. Let $u(t) \in C^2(J, X)$. If $u_t(0) = 0$, then

(1) $\lim_{\alpha\to 1} S^{\alpha}u(t) = -\frac{du(t)}{dt}$

(2) $\lim_{\alpha\to 2^-} S^{\alpha}u(t) = \frac{d^2u(t)}{dt^2}$

Proof. Using Theorem 4.4 and Lemma 2.2, we get

$$\lim_{\alpha\to 1} S^{\alpha}u(t) = \lim_{\alpha\to 1} \frac{d^{\alpha}U(\tau)}{d\tau^{\alpha}} = \frac{dU(\tau)}{d\tau} = -\frac{du(t)}{dt}$$

$$\lim_{\alpha\to 2^-} S^{\alpha}u(t) = \lim_{\alpha\to 2^-} \frac{d^{\alpha}U(\tau)}{d\tau^{\alpha}} = \frac{d^2U(\tau)}{d\tau^2} = \frac{d^2u(t)}{dt^2}$$

where $U(\tau) = u(t)$, $\tau = -t > 0$.

5. NEGATIVE-DIRECTION PROBLEM

Consider the problem

$$(\mathrm{P})^- \qquad \begin{cases} S^{\alpha}u(t) = Au(t), & t > 0, \quad 0 < \alpha \le 2 \\ u(0) = u_0, & u_t(0) = 0 \end{cases}$$

Theorem 5.1. Let $u_0 \in D(A^2)$. If A satisfies (I), then there exists a unique solution $u_{\alpha}(t) \in L_1(J, D(A))$ of the problem $(\mathrm{P})^-$ given by

$$u_{\alpha}(t) = u_0 - \int_0^{-t} r_{\alpha}(s)e^{s}u_0\, ds, \qquad t < 0, \quad 0 < \alpha \le 2 \tag{5.1}$$

which is continuous with respect to $\alpha \in (0, 2]$ and satisfies

(1) $\lim_{\alpha \to 1} u_\alpha(t) = T(-t)u_0 = u_1(t), \qquad t < 0$

(2) $\lim_{\alpha \to 1} S^\alpha u_\alpha(t) = -\frac{d}{dt} u_1, \qquad t < 0$

(3) $\lim_{\alpha \to 2^-} u_\alpha(t) = u_2(t), \qquad t < 0$

where $u_1(t)$ is the solution of the backward problem

$$\frac{du(t)}{dt} + Au(t) = 0, \qquad u(0) = u_0, \quad t < 0 \tag{5.2}$$

and $u_2(t)$ is the solution of the backward problem

$$\frac{d^2u(t)}{dt^2} = Au(t), \qquad u(0) = u_0, \quad u_t(0) = 0, \quad t < 0 \tag{5.3}$$

Proof. Using Theorems 2.1 and 4.4, we get the results.

Now let $X = R^k$ and $u(x, t)$: $R^n \times J \to R^k$; then we have the following definition.

Definition 5.2. The backward fractional diffusion-wave equation is

$$S^\alpha u(x, t) = Au(x, t), \qquad t < 0, \quad 0 < \alpha \le 2 \tag{5.4}$$

and the backward fractional diffusion-wave problem is

$$(\text{D-W})^- \qquad \begin{cases} S^\alpha u(x, t) = Au(x, t), & x \in R^n, \quad t < 0, \quad 0 < \alpha \le 2 \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = 0, & x \in R^n \end{cases}$$

Combining the results of Section 4 and this section, we prove the following theorem:

Theorem 5.3. Let $u_0(x) \in D(A^2)$. If A satisfies (I) with $X = R^k$, then the D-W$^-$ problem has a unique solution $u_\alpha(x, t) \in L_1(J, D(A))$ continuous with respect to $\alpha \in (0, 2]$ and satisfies

(1) $\lim_{\alpha \to 1} u_\alpha(x, t) = T(-t)u_0(x) = u_1(x, t)$

(2) $\lim_{\alpha \to 1} S^\alpha u_\alpha(x, t) = -\frac{\partial u_1(x, t)}{\partial t}$

(3) $$\lim_{\alpha \to 2^-} u_\alpha(x, t) = u_2(x, t)$$

where $u_1(x, t)$ is the solution of the backward problem

$$\frac{\partial u(x, t)}{\partial t} + Au(x, t) = 0, \qquad u(x, 0) = u_0(x), \quad x \in R^n$$

and $u_2(x, t)$ is the solution of the backward wave problem

$$\begin{cases} \dfrac{\partial^2 u(x, t)}{\partial t^2} = Au(x, t), & x \in R^n, \quad t < 0 \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = 0, & x \in R^n \end{cases}$$

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